3 Sobolev Spaces

Exercise 3.1. Let $\Omega \subset \mathbb{R}^d$ be an open set. Prove that

$$\mathrm{Lip}(\Omega) \cap L^\infty(\Omega) \subseteq W^{1,\infty}(\Omega).$$

Moreover, show with a counterexample that when there is no Extension operator between $W^{1,\infty}(\Omega)$ and $W^{1,\infty}(\mathbb{R}^d)$ then the above inclusion is strict, namely there exist functions in $W^{1,\infty}(\Omega)$ which are not Lipschitz.

Exercise 3.2. Let $G : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function such that $G(0) = 0$ and $u \in W_0^{1,p}(\Omega)$ for some $1 \leq p \leq \infty$ and $\Omega \subset \mathbb{R}^d$ open set. Show that $G \circ u \in W_0^{1,p}(\Omega)$.

Exercise 3.3. Let $\Omega \subset \mathbb{R}^2$ be the ball of radius 1 centered at the origin and let $u : \Omega \to \mathbb{R}$ defined as $u(x) = |x|^{-\alpha}$ for $\alpha > 0$. Are there some values of $\alpha > 0$ such that $u \in H^1(\Omega)$? Are there some values of $\alpha > 0$, $p < 2$ such that $u \in W^{1,p}(\Omega)$?

Remark. note that the function u is not continuous, thus in case you find that $u \in W^{1,p}(\Omega)$ for some α and p, it shows the existence of a Sobolev function that is not necessarily continuous. This would not be possible in the 1 dimensional case.

Exercise 3.4. Let $\Omega \subset \mathbb{R}^d$ open and $u \in W^{1,p}(\Omega)$ for some $1 \leq p \leq \infty$. Show that the functions

$$u^+ = \max\{u, 0\}, \qquad u^- = \max\{-u, 0\}, \qquad |u| = u^+ + u^-$$

are in $W^{1,p}(\Omega)$ and their weak derivatives correspond to

$$\frac{\partial u^+}{\partial x_i} = (\chi_{E^+})\frac{\partial u}{\partial x_i}, \qquad \frac{\partial u^-}{\partial x_i} = -(\chi_{E^-})\frac{\partial u}{\partial x_i}, \qquad \frac{\partial |u|}{\partial x_i} = (\chi_{E^+})\frac{\partial u}{\partial x_i} - (\chi_{E^-})\frac{\partial u}{\partial x_i}$$

where $E^\pm = \{x \in \Omega : u(x) \gtrless 0\}$ and χ_E denotes the usual characteristic function of the set E. Finally, for every $M > 0$, deduce that the truncated function

$$T_M u = \max\big\{\min\{u, M\}, -M\big\}$$

is also in $W^{1,p}(\Omega)$.

Exercise 3.5. Let $1 \leq p < \infty$, $u \in W^{1,p}(\mathbb{R}^d)$ and $\{e_i\}_{i=1}^d$ be the canonical orthonormal basis of $\mathbb{R}^d$. Let $h > 0$ and set

$$\partial_i^h u(x) = \frac{1}{h}\big(u(x + h\,e_i) - u(x)\big) \quad \text{whenever } u \text{ is well defined at the points } x,\ x + he_i.$$

Observe that $\partial_i^h u$ is well defined a.e. in $\mathbb{R}^d$ and show that $\partial_i^h u(x) \to \partial_{x_i} u$ strongly in $L^p(\mathbb{R}^d)$ as $h \to 0$.